The Intersection of Circles and the Intersection of Spheres.

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1. In this paper I propose to discuss the various problems of the Intersections of Circles and the Intersections of Spheres.

The first problem is to draw a circle which shall make a certain given angle with three given circles. The next is to draw a sphere which shall cut each of four given spheres at a given angle. Afterwards I shall take up the two problems (proposed by Steiner, 1st Vol. of Crelle, 1826, page 163) to draw a circle which shall cut four given circles at the same angle (said angle being unknown): also to describe a sphere which shall cut each of five given spheres under the same angle (said angle being unknown). All these problems are solved geometrically.

2. In each case I shall give the number of solutions. It has been long known that to draw a circle tangent to or intersecting three given circles at the same angle, there are 8 solutions; also that there are 16 solutions to the question to draw a sphere intersecting or touching four given spheres at a given angle. But I think it has not been known that there are 96 solutions to the question to draw a circle cutting four given circles at the same angle, or that there are 640 solutions to the problem to draw a sphere to cut five given spheres at equal angles. But in many of them the intersections, either in circles or spheres, will be imaginary.

3. I expect to show that the central geometrical principle from which all these solutions are evolved is that of the radical centre, or centre of the orthogonal circle, which was the basis of the solution of the Problem of the Tangencies of Circles and of Spheres given by me, and published in 1855 in the 8th Volume of the Smithsonian Contributions. In this dissertation the tangency will be regarded as the case in which the intersection is at an angle of 0 or 180°

In general we can say that two circles make with each other an angle β, viz., the angle which the tangents at the point of intersection make with each

other, or the supplement of the angle which the two radii drawn to the points of intersection, make with each other.

4. In discussing the intersection of circles, there may in a certain sense be said to be ten different cases (the number of the cases propounded by Appolonius in reference to tangencies); but all of them are really embraced in the single one, to draw a circle cutting each of three given circles at a given angle: some of those circles can be reduced to points, *i. e.* circles with radius 0, or to right lines, *i. e.* circles with radii ∞, and in this way 10 sub-cases arise. The table giving the various questions in tangencies, at page 4 of my Smithsonian memoir, would answer for the intersections, substituting for the word "tangent," a certain angle β, &c. And as in that memoir, Problem 4 is the central problem containing the gist of the whole, so Problem 4 in the theory of intersections bears a like relation to the rest: this is to draw a circle through two given points cutting a given circle at a given angle.

5. I premise that I shall consider as known to the reader the principles of the polar line and polar plane; of the radical axis, centres or planes of similitude, the radical centre and orthogonal circles or spheres, as given in the so-called modern geometry. But I shall be compelled to reproduce a few of them of most frequent use in this paper, but only so far as may be necessary to throw light on their subsequent development.

6. I have found the most lucid way to consider, and to construct, the radical axis of two given circles, is to draw any circle cutting both; the right lines joining the two points of intersection in each, those two lines will intersect each other in the radical axis; then by another secant circle finding in like manner another such point, the radical axis is determined. If from any point of this axis as a centre, and the tangent line as a radius, a circle be drawn, it will cut each of the two given circles orthogonally.

It is of frequent use to remember, that if we mark the two points in which one of said orthogonal circles cuts the line joining their centres, *all* circles drawn through those two points will be orthogonal to the two given circles and have their centres on the radical axis. Also, if it is proposed to find a circle orthogonal to a given circle and having its centre on a given line, we may proceed as follows: let fall a perpendicular from the centre of the given circle upon the line; draw at random from any point of it, a tangent to the given circle, and use it as a radius to draw an orthogonal circle thereto: it will intersect the perpendicular in two points. *Any* circle whatever drawn through those points will be orthogonal to the given circle.

Thus in Figure 1, RR' is the radical axis of circles A and B. From any point of R draw a tangent RM, and with it as a radius describe a circle orthogonal to them. It will cut the line AB at Q and Q'. *Any* circle through Q and Q' will be orthogonal to circles A and B; as $AQ \times AQ' = AM^2$, whatever may be the position of M in the circumference.

If as in Figure 2 it is required to draw through Q a circle orthogonal to circle A: join Q with the centre A; find the point Q' on line joining the centre A such that $AQ \times AQ' = AE^2$. *All* circles through Q and Q' will be orthogonal to the given circle. Or given a right line RR', to draw a circle orthogonal to circle A, having its centre on said line; draw *any* circle orthogonal to circle A and having its centre on RR', and mark the points Q and Q' in which it cuts AK drawn from the centre perpendicular to RR'. Any circle as before through Q and Q' will fulfil the conditions.

7. If as in Figure 3 it is required to draw a circle orthogonal to circle A, touching the circle D, and having its centre on a given line FT: find as in the last article the two points Q and Q' through which all circles must pass orthogonal to circle A and having their centres on FT. Then by Problem 4, (Tangencies of Circles,) through the two points Q and Q' draw a circle tangent to circle D. It will also be orthogonal to circle A. Draw any secant circle as QHQ' to circle D. The chord $H'H$ will give O the "centre of converging chords." Then draw from O to circle D two tangent lines OM, OM'. There are thus two such circles having their points of contact at M and M', and their centres at F and F'.

8. As in the preceding article I shall have frequent occasion to speak of the centre of converging chords, or to refer to "the principle of converging chords"; by this I mean only the well-known principle of the radical centre. The latter term is applied generally to three circles, but the former to a system or group of any number of circles. The three radical axes of three given circles, it is well known, meet in the same point, called the radical centre, or the centre of the circle orthogonal to the three given ones.

In Figure 4, let the three circles first considered be the circles C, ABF and ABH. Their three radical axes are the lines AB, FF', HH', all uniting at O, their radical centre. But if a fourth or any secant circle, as $ABII'$, is added: the chord $I'I$ will also pass through the same point O, which I call the centre of converging chords. The principle of converging chords can be enunciated as follows: If a fixed circle is cut by any circle or system of circles, which pass

through two given points in the plane of the given circle, the chords pass through a fixed point on the line passing through the two points. Or to use language still more general: Any system of circles having a common radical axis (whether they intersect or not) will cut a fixed circle in chords converging towards a fixed point on such axis. In Figures 3 and 4 they intersect. But in Figure 5 suppose the case in which they do not intersect—as the system of circles FF', GG', HH', orthogonal to a fixed circle AQ, having their centres on QQ', and having AO for a common radical axis. Mark the intersections of this system with a given circle C; the chords $D'D$, $E'E$, $L'L$ will all converge to a point O on the radical axis, the centre of converging chords, in reference to circle C, of the whole series. It will be noted that in this figure the whole series of circles passing through QQ' having the line QQ' for a radical axis, are orthogonal to each circle in the other series above named having AO at right angles to it, for a radical axis.

9. If then the problem is proposed to draw one of this last series, viz. a circle tangent to a given circle C in Figure 5, having its centre on a given line QQ', and orthogonal to circle AQQ': draw at will any such orthogonal circle as HH', join the points D and D' in which it intersects said circle by a line meeting the radical axis in O. This will be the "centre of converging chords." From O draw tangents to the circle C, these will determine W and W', the two points of contact of the pair of required circles. The circles X and Y in the figure will be each tangential to C and orthogonal to A.

10. The well-known principle of the polar line and its pole is exhibited in Figure 6. If through any point E, interior or exterior to circle A, any number of chords (or secants) be drawn, and from the extremities tangents be drawn, they will all unite in the same line PX. For draw one chord BB' through E as well as $H'H$, the shortest chord through E, and their pairs of tangents, the latter uniting at P, the former at X, $AD \times AX = R^2 = AE \times AP$. Therefore the triangles ADE and APX are similar, and the angle APX is a right angle, whatever may be the position of X. Thus if chord II' be drawn, the point X' will be on the same line PX. A glance at the diagram will show that XX' is also the locus of circles orthogonal to circle A passing through a given point Q, as in Figure 2. All such circles will pass through Q and Q', and the chords can be regarded as "the converging chords" of the circles having E for its centre. Thus the principle of the pole and its polar line may be regarded as derived from that of the converging chords, or of the radical centre.

11. I will now proceed to discuss the question referred to in Article 4, as follows: *To draw a circle through two given points to cut a given circle at a given angle* β.

The solution is reduced to a question in tangency, and to finding a circle orthogonal to one auxiliary circle and tangent to another.* In Figure 7, through the given points D and E to draw a circle cutting the given circle C at a given angle β.

Draw the triangle CMN with angle $CMN = \beta$, $MN = \cos\beta$. With $CN =$ sine β, draw a circle concentric to the given circle. Through E, one of the given points, draw the circle EI with a radius equal to $MN =$ cosine of β. Find as in the last article the circle orthogonal to auxiliary circle CN and tangent to auxiliary circle EI, and having its centre on the line BA perpendicular to DE through its middle point. There are two such circles XH and YF. Circle X is orthogonal at H to circle CH. Mark the point X' in which the tangent line XH cuts the given circle. $EI = X'H = MN = \cos\beta$. Therefore $XE = XX'$, and the point X is the centre of the required circle making an angle $CX'H = \beta$ with the given circle. In like manner circle Y is another answering the conditions, as $FY'' = MN$. If $\beta = 0$ or the required circle is to be a tangent to the given circle, find S, the "centre of converging chords," and draw two tangent lines to circle C; they will give the two points of contact K and K' of the pair of tangent circles, G and G' being their centres.

12. I will now explain some properties of the orthogonal circle not belonging immediately to the discussion, but which will be hereafter referred to. Join the two points of contact K', K, and from A where the line cuts BX, draw two tangents AP and AP'. $AK \times AK' = AP^2$, and $SP \times SP' = SK^2$ (or AKK' is the polar line of S as pole in reference to circle C). The circle DPP' is orthogonal to circle C, or answers to the case in which β is $90°$.

13. This orthogonal circle bisects the angle which the two required circles DEX', DEY', make with each other. But I will first show that it bisects the angle which the two tangent circles DEK, DEK' make with each other at the point E. Extend $K'KA$ to R. Extend CK' to G'', making $K'G'' = GK$. Join GR and GG'', EG, EA and EX. GG'' is parallel to AK', and the triangle $G'AK'$ similar to $G'GG''$, and

$$G'K' : K'G'' :: G'A : AG. \quad \text{But } K'G'' = KG = GE \text{ and } G'K' = G'E.$$

* *Note by Prof. Cayley.*—Thus it can be shown that if two circles, radii R and C cut at angle β, then the concentric circles, radii $R + C\cos\beta$ and $C\sin\beta$ will cut at right angles.

Therefore $G'E : EG :: G'A : AG$. Thus in the triangle GEG' as the line EA divides the base so that its segments are proportional to the other two sides, it follows that the radius EA of the orthogonal circle bisects the angle GEG', or the angle which the two tangent circles make with each other.

Also A, the centre of the orthogonal circle, is a centre of similitude of the two tangent circles. The angles ARG, AKG, CKK', $CK'K$ being all equal, the radii $G'K'$ and GR are parallel, and therefore the line RKK' passes through the "external centre of similitude."

By a similar chain of reasoning it will easily be seen that AE bisects the angle XEY which the two required circles (cutting the given circle at angle β) make with each other, and the lines $AX'Y'$, $AX''Y''$ are straight lines, and A the centre of the orthogonal circle is also the "internal centre of similitude" of the two required circles X, Y.

Also $AE^2 = AP^2 = AK \times AK' = AX' \times AY'$; or the radius of the orthogonal circle is a mean proportional to the distance of its centre from the pair of tangent circles, or from the pair of required secant circles.

Suppose the two tangent circles DEK, DEK' to remain fixed, and the circle C to change, remaining tangent to them: the right line joining the points of contact will continue to pass through A. Draw at will any right line through A, mark the points in which it cuts the two tangent circles and the orthogonal circle, draw tangents at each of those points, the two first will make equal angles with the last, or the curve of the orthogonal circle bisects the angle which at those points the curves of the two tangent circles make with each other.

It will be found that in the whole series of problems regarding intersections, the circumference and the centre of the orthogonal circle enjoy similar properties to those above explained.

14. I will now investigate the problem: *Through a given point to draw a circle cutting two given circles at a given angle β.*

In Figure 8 let P be the given point and A and B the given circles. Let TO be the radical axis of those circles; Q and Q' the points (as in Article 6) through which all circles orthogonal to those two circles must pass. Through P, Q, Q' pass a circle, it will be the orthogonal circle through the given point to those circles, having of course its centre O on the radical axis. Prolong the radii BE and AK to meet at Y. The triangles OYK, OYE are equal. Therefore the angles AKR' and BRE are equal, and the radii BR, AK are parallel. There-

fore the lines EK or $E'K'$ must pass through S the "external centre of similitude" of the two given circles. The point P' in which SP cuts the orthogonal circle is determined by means of the relations $SN \times SN' = SL \times SL' = SP \times SP' = SE \times SK = SE' \times SK'$. Then by Article 11, through the points P and P' draw a circle making the required angle with circle A, it will make the same angle with circle B. $PP'HD$ with centre at (1) is one of those required circles, $PP'VM$ another, with centre at (2). Join HD and produce it to W on circle A. Join (1), the centre of first required circle, with Y', the point of intersection of the two radii BH, AD. The triangle (1) HD is isosceles. The angle (1) $DY' =$ (1) $HY' = \beta$. Therefore the triangle $Y'HD$ is isosceles. And the angle $Y'DH = Y'HD = AWD$. Therefore the radii BH and AW are parallel, and the line HW must pass through S, the "external centre of similitude" of the two circles.

When the angle β is zero (the case of tangency), join the points of contact with each circle of the pair touching each of the given circles in the *same* manner, the line will pass through the "external centre of similitude." The other pair of tangent circles will touch the given circle in a *different* manner, and the line joining their points of contact will pass through the "internal centre of similitude."

Join P with S' the "internal centre of similitude," and mark P'' the point in which PS' cuts the orthogonal circle. Then through P and P'' (as in Article 11) pass a circle cutting circle A at the given angle β, it will also cut circle B at the same angle. There are two such circles, and thus there are four in all fulfilling the conditions of the problem. The centres of these two circles are at (3) and (4) in Figure 8.

Join the centres of the first two (1) and (2), the line must pass through O the centre of the orthogonal circle, and be perpendicular to the axis of similitude SPP'. Join the centres (3) and (4) and the line passes through the same centre and is perpendicular to the axis of similitude $PS'P''$. This will be found to be a general rule, extending to the tangent circles. To three given circles it is well known there are (Article 18) four axes of similitude passing through their centres of similitude in groups of three each. In this case, where one of the three circles is the point P (or a circle of radius zero), there are but two axes of similitude, as above indicated.

15. Thus the four required circles through the given point intersect each other in pairs in the circumference of the orthogonal circle, at the points in

which the two axes of similitude cut it. And it will readily be seen by a method similar to that employed in Article 13, that the orthogonal circle bisects the angles which each pair of the required secant (or tangent) circles make with each other. Through the centre of the orthogonal circle (or radical centre) draw at will any line cutting the circumferences of each pair, and mark the distances to each circumference along that line from the centre—the radius of the orthogonal circle is a mean proportional to these distances.

The radical centre is also *one* of the centres of similitude of each pair of the required circles. Call $\frac{2}{4}$ the point in which circles 2 and 4 intersect, and $\frac{1}{3}$ the point in which circles 1 and 3 intersect; the right line $\frac{2}{4}\,\frac{1}{3}$ joining those two points passes through the radical centre, and the angle at which 2 and 4 intersect, it is easily proved, is the same as the angle at which 1 and 3 intersect. So also the right line $\frac{2}{3}\,\frac{1}{4}$ will pass through the same centre, and the angle at which 2 and 3 intersect is the same as the angle at which 1 and 4 intersect.

16. It should be noted that any circle through P and P' will cut the two given circles at the same angle, and that a mate to it will be found cutting them at a like angle. The same is true of any circle through P and P''. If it should not meet the circles, the intersections will be imaginary, and each pair will be situated in a similar manner in reference to each of the given circles, and the orthogonal circle will bisect the angle each of a pair make with one another.

17. It should also be noted that if from S, the external centre of similitude of the two given circles, a tangent line be drawn to one of the required circles, as SZ, we shall have $SZ^2 = SP \times SP' = SD' \times SH' = SE \times SK$. Hence the said circle will be orthogonal to the pair of required circles cutting or tangent to the given circles in a similar manner. [This is called by Steiner in the 1st Volume of Crelle the "power" circle, "Puissance" circle, and is useful in certain exceptional cases.]

In Figure 11 draw a circle orthogonal to circles A and B, cutting them at P and P', and with any point X on the radical axis as a centre, $P'P$ will, we have seen, pass through S the external centre of similitude. The "power" circle ZZ' will have a radius SZ tangential to PP', and $SZ^2 = SP \times SP' = SO \times SO' = ST \times ST'$. Thus XY is a common radical axis to circles A and B and also to the "power" circle. This power circle is orthogonal to the whole

series of circles either tangent to or making equal angles with A and B. We have before seen how the orthogonal circle bisects the angles which two circles make with each other when they intersect. But it may also in a certain sense be said to bisect the angle which circles A and B make with each other *though they do not intersect.* This might be named their *angle of approach.* Draw any straight line through S, the centre of the power circle, cutting the circles A and B, the radius of the power circle is a mean proportional to the distances along that line to each circumference.

But this power circle bisects the angle of approach in another sense (as in Article 11.) Draw any line as SOO' through S, cutting A and B at O and O' and the power circle at D. Draw tangents at O, O' and D. Those from O and O' will unite at Y on the radical axis. Join FY. HDH' is parallel to FY. The angle $FYO = FYO' = DHO = DKO' = \frac{1}{2}\, OYO'$. Or the curves at O and O' make equal angles with the curve at D.

18. We shall pass over the problems "to draw a circle which shall cut a given circle and two given right lines at the same angle," and, "to draw a circle which shall cut a given line and two given circles at the same angle," which are readily solved by like principles, and pass to the 10th and last problem in the series, which indeed is *the* general problem of this investigation, and comprehends all.

To draw a circle which shall cut each of three given circles at a given angle.

Let A, B, C, Figure 10, be the three given circles. Find the three external centres of similitude N', M', S', and the three internal centres of similitude N, M, S of said circles. The three external are in one straight line, called the external axes of similitude of the three circles, and three other axes are found by joining each of the external centres of similitude with the internal centres of similitude. There are thus four axes of similitude to the three circles. (See *Rouché et Comberousse, Traité de Géometrie élémentaire,* Art. 389.) As in Article 13 it will be found that each of these four axes of similitude is the radical axis of a pair of the required circles (whether secant or tangent) or eight in all. In Figure 10 draw the circle whose centre is O orthogonal to the three given circles A, B, and C. Call the centres of the required circles (1) (2) (3) (4) (5) (6) (7) (8). Let $\frac{7}{8}\ \frac{7}{8}$ be the points in which circles 7 and 8 intersect, and so

on; the line $\frac{7}{8}\,\frac{7}{8}$ will be one of the axes of similitude; the lines $\frac{5}{6}\,\frac{5}{6}$, $\frac{3}{4}\,\frac{3}{4}$, and $N'M'S'$ being the other axes of similitude. This is supposing the circles 1 and 2 do not intersect as in this diagram; the case in which they do intersect will be treated separately in the next article.

Thus the solution of this problem consists in finding the two points in which any axis of similitude cuts the orthogonal circle; through those two points draw, as in Article 14, a circle cutting one of the given circles at the given angle β (or zero in case of tangency), and it will cut the two other given circles at the same angle.* Indeed, *any* circle through those two points will cut the three given circles at the same angle, or if it chances not to intersect it (the intersection being imaginary) it will (Article 16) be situated in a similar manner towards each. Thus these points $\frac{1}{2}\,\frac{1}{2}$, $\frac{3}{4}\,\frac{3}{4}$, $\frac{5}{6}\,\frac{5}{6}$, $\frac{7}{8}\,\frac{7}{8}$ might be called the key points of the problem. They are referred to by Poncelet and Salmon for other purposes, and are called by them "limiting points."

19. We will now consider the exceptional case in which the two required circles as 1 and 2 do not intersect in the circumference of the orthogonal circle, but will have as a radical axis the external axis of similitude, sometimes named the axis of "direct similitude." In Figure 12, let A, B, C be the given circles, NMS the external axis of similitude, FEE' the orthogonal circle. Through F the radical centre let fall FF' a perpendicular to said axis, it must contain the centres of the two required circles. In circles B and C draw auxiliary circles CH', BL', with radii equal to $\sin\beta$ in each circle. That is, make the angle $CHH'=\beta$. Lay off $HI=\cos\beta$ in circle B, and draw the auxiliary circle CI, with centre at C.

Find the points Q and Q' through which (see Article 7) all circles must pass orthogonal to circle BL' and having their centres on FF'. Through Q as a centre draw the auxiliary circle QD' with a radius equal to $H'I=$ cosine of CIH', a known angle.

Draw a circle, as in Article 9, having its centre on FF', tangent to circle QD' and orthogonal to circle CH', intersecting the latter at W. Its centre X is the centre of the required circle. For, from X draw the right line XW tangent

Note by Prof. Cayley.—The theorem seems well worthy of an independent statement, viz.: Take any circle O and three circles A, B, C, each cutting O at right angles. Take any one of the four axes of similitude meeting O in two points X and Y (these points are imaginary for the exterior axes, but real for each of the other three), then the theorem is that any circle whatever through X and Y cuts each of the circles A, B, C at the same angle β.

to circle CH', and $XL'L$ tangent to circle BL'; circle $QQ'L'$ will be orthogonal at L' to circle BL'. Mark the points Z and L''. These are points in the required circle; for

$$QD' = WO'' = H'I = LL'$$
$$ZW = HH' = \cos.\ \beta \text{ in circle } C.$$
$$L'L'' = ZO' = HI = \cos.\ \beta \text{ in circle } B. \quad Q.\ E.\ D.$$

Find the other circle having its centre on FF' tangent to circle Q, and orthogonal to circle BL'. The centre of this circle Y is the centre of the mate to KZ, cutting the given circle at the required angle.

Let KK', RR' be the arcs of intersection of the two required circles with circle A. The two lines FKR, $FK'R'$ will be straight lines. Mark the point P in which the line EE' meets $F'S$. P is "the centre of converging chords" of all circles having their centres on FF' and the radical axis $F'S$. Therefore $K'K$ and $R'R$ will pass through P. Draw the two tangents from P to circle A. The points of contact will be the points of contact of the pair of circles tangent to the three given circles, or the case of $\beta = 0$.

More generally draw at will through P any straight line cutting circle A. Mark the two points of intersection. Pass through these two points a circle having its centre on FF', it will cut the three given circles at the same angle, but not necessarily at the given angle β. Again, draw, with S (the external centre of similitude of A and C) as a centre, the "power circle" $P'T$, which is orthogonal to all the circles cutting A and C at the same angle, and in like manner as in Article 17. Then if with *any* point X of FF' as a centre and a radius equal to XX', drawn tangent to this "power circle," a circle be drawn, it will cut the three given circles at equal angles.

20. I append a *resumé* of some of the curious relations of the required circles to each other and to the orthogonal circle and the radical centre. Although not belonging strictly to the problem, they have arisen from time to time in the course of the investigation and cannot well be omitted. The first three heads in this recapitulation have been well known for many years, but the remaining four heads set forth new properties, so far as I can ascertain. After long and patient research I find no allusion to the circumference of the orthogonal circle bisecting the angle between the pairs of secant or tangent circles, or to its centre being a point from which radiate such a multitude of lines.

RECAPITULATION OF THE THEOREMS WHICH OCCUR IN THE SOLUTION OF THE PROBLEM TO DRAW ALL THE CIRCLES CUTTING THREE GIVEN CIRCLES AT ANY GIVEN ANGLE.

1st. The required secant circles are eight in number, distributed in pairs, the radical axis of each pair being one of the "axes of similitude" of the three given circles.

2d. Each pair intersect in the circumference of the orthogonal circle; or if a pair do not intersect, the orthogonal circle has with each of them the same radical axis—one of the axes of similitude.

3d. The line joining the centres of each pair of the required circles passes through the radical centre. And the radical centre is the "internal centre of similitude" of each pair of the required circles, and thus in the case of tangency the line joining the points of contact passes through the radical centre.*

4th. If a pair of the required circles intersect in the circumference of the orthogonal circle, that circumference bisects the angle which they make with each other. If they do not intersect, the orthogonal circle bisects what may be called their *angle of approach;* the radius of the orthogonal circle is a mean proportional between the distances from its centre on any straight line to the circumferences of the circles; and the orthogonal circle (Article 17) makes equal angles with the curves at the points of intersection of the line.

5th. In any pair of the required circles join the corresponding points of their intersections with any one of the given circles; each line of junction will pass through the radical centre. Furthermore, the radius of the orthogonal circle is a mean proportional between the distances along those lines from the radical centre to each of these points of intersection, or to the points of contact of any pair of the required circles, in the case of tangency.

6th. Join the corresponding points of intersection of any one secant circle with any pair of the given circles, the line of junction will pass through one of the centres of similitude of that pair (Article 14).

*This last fact is the one employed in the elegant solution of the problem of Tangencies by Gergonne in Volume 4, Annales de Mathématiques, 1814. This, concisely stated, is as follows. Find in each of the given circles the pole of each axis of similitude. Join it with the radical centre, the joining line will pass through a pair of the required points of contact in each circle. This solution is justly admired. But it is only one of the last and crowning facts of the development. The writer aims at the opposite method, to commence at the beginning and evolve the entire series from an elementary geometrical principle.

7th. The pairs of required circles above referred to are those having the same radical axis, viz. one of the axes of similitude. Find the *other* intersections of the required circles, those not in the circumference of the orthogonal circle. If the intersection farthest from the radical centre is joined with the intersection nearest, the line will pass through the radical centre. In that way we shall find 24 such lines passing through the radical centre, and these circles in pairs intersect each other in equal angles (Article 15).

No. of Group.	24 lines which pass through the centre of the orthogonal circle.				
I	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{4}$	
	$\frac{2}{4}$	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{1}{3}$	
II	$\frac{2}{5}$	$\frac{1}{6}$	$\frac{2}{5}$	$\frac{1}{6}$	
	$\frac{2}{6}$	$\frac{1}{5}$	$\frac{2}{6}$	$\frac{1}{5}$	
III	$\frac{2}{7}$	$\frac{1}{8}$	$\frac{2}{7}$	$\frac{1}{8}$	Circles 2 and 3 intersect at the same angle as circles 1 and 4, and so on through the list.
	$\frac{2}{8}$	$\frac{1}{7}$	$\frac{2}{8}$	$\frac{1}{7}$	
IV	$\frac{3}{5}$	$\frac{4}{6}$	$\frac{4}{6}$	$\frac{3}{5}$	
	$\frac{3}{6}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{3}{6}$	
V	$\frac{4}{7}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{4}{7}$	
	$\frac{4}{8}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{4}{8}$	
VI	$\frac{6}{7}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{6}{7}$	
	$\frac{6}{8}$	$\frac{5}{7}$	$\frac{5}{7}$	$\frac{6}{8}$	

THE INTERSECTION OF SPHERES.

21. It will be found that the problems in relation to spheres can be solved by a similar mode of investigation to that which has been followed in those concerning circles. By altering the table on page 10 in the Memoir by the writer in the 8th Volume of Smithsonian Contributions, substituting for "tangent to" the words "*making a given angle β with,*" it will be seen that there are fifteen such problems for the intersections as there are for the tangencies of spheres. Of course two spheres intersecting make an angle β with each other when the right-lined elements of the cones drawn tangent to each sphere through the common circle of intersection, make that angle with each other (or the supplement to it).

If two spheres intersect, their radical plane is the plane of the small circle in which they intersect. Draw pairs of spheres tangent in every conceivable manner to them. It will be found that the radical plane of each pair will pass through the *external* centre of similitude if they are tangent in a *similar* manner, that is, both concave or both convex towards the given spheres. If they are tangent in a *different* manner, their radical plane will pass through the *internal* centre of similitude of the pair of spheres. The same is true of the pairs of secant spheres in this problem, being distributed in like manner in reference to the centres of similitude.

22. *To draw a sphere through three given points and making a given angle with a given sphere.*

Pass a circle through the three points, it will form a small circle of the required sphere. From the centre of the sphere let fall a perpendicular to the plane of said circle. Pass through this perpendicular and the centre of the circle a plane. It will cut the circumference of the small circle in two points, and it will cut the given sphere in a great circle. Then by Article 11, through these two points draw a circle making the required angle β with said great circle, it will form a great circle of the required sphere. Thus two such can be found, except when the given angle is a right angle, in which case there is but one.

23. The radical plane of two spheres is the plane from any point of which if tangent lines are drawn to each sphere they will be equal. Any such point is the vertex of a pair of cones tangent to each sphere, having elements of equal length, and it is the centre of a sphere orthogonal to the pair of spheres. And as in Circles—see Figure 1—two points Q and Q' can be found on the line

joining the centres, through which all spheres must pass which are orthogonal to the pair of spheres.

When three spheres are given, as A, B and C (Figure 9), whose centres are in the plane of the paper, the three radical planes will unite at O the radical centre of these great circles of the spheres. Find QQ', HH', EE', the points above referred to. The perpendicular through O to the plane of the paper may be called the common radical axis of the three spheres. Any sphere orthogonal to the three spheres must have its centre on this perpendicular, and pass through the circle drawn through the six points QQ', HH', EE'. Therefore if at will *any* sphere whatever is passed through said circle, it must be orthogonal to the three spheres.

24. *To find the sphere orthogonal to four spheres.*

If a fourth sphere D is given whose centre, in Figure 9, is above the plane of the paper, draw the radical plane between A and D, the point in which said plane cuts the radical axis through O, above described, is the centre of the required orthogonal sphere, or the radical centre of the four spheres. The six radical planes which can be drawn to the four given spheres in pairs will all pass through the radical centre. There are sixteen axes of similitude to the four spheres, four external, twelve internal. There are eight planes of similitude, one external containing the four external axes of similitude.

25. *To draw a sphere which shall cut four given spheres at a given angle.*

It will be found that there are sixteen solutions to this question, as in the Tangency of Spheres, distributed in pairs, each of the eight planes of similitude being the radical plane of a pair. Thus the required spheres intersect each other in pairs in the surface of the orthogonal sphere, the small circle of intersection being their radical plane, which must be one of the planes of similitude of the four given spheres. If any pair do not intersect they still have one of the planes of similitude for a radical plane, and the solution will be an exceptional case, as in Article 19 in Intersection of Circles, and solved in an analogous manner.

Thus to obtain one of the required spheres, find graphically, as in descriptive geometry, the small circle in which one of the planes of similitude intersects the orthogonal sphere. Then proceed as in Article 22: pass a sphere through the circle which will cut one of the given spheres at the given angle β; it will cut the other three spheres at the same angle.

26. The circle referred to has this remarkable property. Any sphere whatever passed through it will cut the four given spheres in the same angle. Or if it does not meet them the intersection becomes imaginary, but it is (as in Circles, Article 16) situated in a similar manner towards each of the given spheres.

27. I might append a complete recapitulation of the peculiar relations of the orthogonal sphere and its centre to the required spheres similar to that given for Circles at Article 20. I subjoin a portion of the same.

If a pair of the required spheres intersect in the surface of the orthogonal sphere, the latter bisects the angle which they make with each other. If they do not intersect, the orthogonal sphere bisects their "angle of approach": also its radius is a mean proportional to the distances from its centre on any straight line to the surfaces of the pair of spheres, and the surface of the orthogonal sphere makes equal angles with their surfaces, estimated at the points in which the straight line meets them.

28. With reference to the number of lines passing through the radical centre of the four spheres:—As regards a pair of required spheres intersecting in the surface of the orthogonal sphere, the line joining the centres of the pair will pass through the radical centre and be perpendicular to the corresponding plane of similitude. If tangential, the line joining the points of contact of the pair with any one of the given spheres will pass through the radical centre. If secant, find the two circles of intersection of such pair with any one of the given spheres, from the radical centre as a vertex pass a cone through the nearest circle, its surface produced will pass through the farthest circle.

29. Again, the intersections of the required spheres (whether tangent or secant) with each other which do not occur in the surface of the orthogonal sphere are distributed in pairs and will obey a law similar to that described in the case of Circles, Article 20, or paragraph 7 in the Recapitulation. Draw a cone from the radical centre as a vertex to the nearest circle of intersection. Its surface produced will pass through the circle of intersection farthest from that centre. And the angle of intersection of the first pair will be equal to the angle of intersection of the second pair of the required spheres.

30. In conclusion it may be stated that articles 11, 18 and 25 set forth the main principles of the generalized theory. Article 11 shows that the solution of the questions in intersections is reduced to one in tangencies, and in the case

of orthogonal circles, all evolved from the principle of the converging chords, or rather of the radical centre.*

31. *To draw a circle which shall cut each of four given circles at the same angle.*

We found in the problem, Article 18, to draw a circle to cut each of three given circles at the same angle, that after finding the two points as K^1 and K^2 in which one of the axes of similitude cuts the orthogonal circle, *any* circle drawn through those two points (Figure 13) will cut the three given circles A, B and C at the same angle. Introduce in the figure a fourth circle D. Find the external and internal centres of similitude of B and D and call them N and N^1. First proceed as if to find in reference to B and D (as in Article 14) a point conjugate to K^1, through which all circles which can be drawn will cut B and D at equal angles. Join K^1 with N, and find the point R on NK^1 such that $NK^1 \times NR = NL \times NL^1$, a constant quantity. Thus those three points K^1, K^2 and R will enjoy these properties—all circles through K^1 and K^2 will intersect circles B and C at equal angles, and all circles through K^1 and R will intersect B and D at equal angles. Hence if we draw through K^1, K^2 and R a circle, it must cut the *four* given circles at one and the same angle.

In like manner join K^1 with N^1 the internal centre of similitude of B and D, and find R^1 a point conjugate to K^1 such that $N^1K^1 \times N^1R^1 = N^1L \times N^1T$. A circle drawn through the three points K^1, K^2 and R^1 will also cut the four circles at the same angle. Thus two solutions are found by the use of the *key* points K^1, K^2. [It must be remarked that if K^2 had been joined with N and afterwards with N^1 in like process, the same identical pair of required circles would be reached, not two new ones.]

In like manner two solutions can be obtained by combining the key points K^1, K^2 with the two centres of similitude of C and D, and two more with those of A and D, or six in all by the use of those points, as shown in the table following:

*It is proper to add that the writer, in the 4th Volume of Johnson's Cyclopedia (published in 1878), under the head of "Tangencies," gave a history of the problem of the Tangencies. The principal addition which now should be made to it is to refer to the important paper by Plücker in Vol. 18 of Gergonne's Annales de Mathématiques, 1827, page 29. Also an article by Poncelet in Vol. 11 of same publication. The title of Plücker is "Memoir upon the contacts and the intersections of circles." It contains a very curious dissertation, but has not anticipated the writer, in the present generalization. In the intersections he takes an entirely different process. My paper in the Smithsonian Contributions, 1855, on "Spheres," as indeed the whole treatment of the subject is believed to be entirely novel. But Plücker in "Circles" gives an analytical solution founded on increasing the radius of each given circle by a certain quantity.

Let M and M^1 be the centres of similitude of circles C and D, P and P^1 of A and D.

By combining K and K^1 with	N and N^1	 2	solutions.
" " "	M and M^1	 2	"
" " "	P and P^1	 2	"
		6	

But there are four axes of similitude of A, B and C, and thus four pairs of key points, or points in which each axis will cut the orthogonal circle. We have called K^1 and K^2 the first pair, and will name the other pairs K^3 and K^4, K^5 and K^6, K^7 and K^8. Treating each pair of key points as above in combination with the centres of similitude of A, B and C respectively as paired with D, there will be 6 solutions for each of the four pairs or 24. Tabulated as follows:

The centres of similitude of A, B and C as paired with D joined with	K^1, K^2	. . . 6
" " " "	K^3, K^4	. . . 6
" " " "	K^5, K^6	. . . 6
" " " "	K^7, K^8	. . . 6
		24

But thus far we have only investigated the number of solutions when A, B and C are combined with D. Combine by the same process any three with a fourth and 24 solutions will be obtained. There are four axes of similitude of *each* group of three of the given circles, each giving a pair of key points to be combined as above with the centres of similitude of a fourth circle. Accordingly there results from

			No. of Solutions.
A, B, C	combined with two centres of similitude of each combined with	D . . .	24
A, B, D	" " " "	C . . .	24
A, C, D	" " " "	B . . .	24
B, C, D	" " " "	A . . .	24
		Total	96

There are thus in general ninety-six distinct answers to the problem of finding a circle cutting four given circles at the same angle, which, of course, will usually be different for each solution.

But it is to be noted that some of the intersections will be imaginary; as Salmon (in Conic Sections, page 105) discusses the case of an imaginary intersection of two circles, but that the pair continue after separated to have a common radical axis. But the required circle in such cases would still be situated (Article 16) in a similar manner towards each of the four given circles.

32. *To draw a sphere which shall cut each of five given spheres, A, B, C, D and E, at the same angle (said angle being unknown).*

I need hardly premise that a sphere becomes known when two of its small circles are found, as its centre can be obtained by the intersection of two perpendiculars to the planes of the circles, erected from their centres.

Therefore find the sphere orthogonal to A, B, C and D, and also one of their planes of similitude: it will as explained above cut the orthogonal sphere in a small circle, which (Article 26) enjoys the property that any sphere drawn through it will cut those four spheres at the same angle.

In like manner find the sphere orthogonal to A, B, C and E and also one of their planes of similitude, and the small circle in which it cuts said orthogonal sphere. As any sphere passed through this last circle will cut each of A, B, C and E at the same angle, if a sphere is drawn through these two small circles it must be one of the required spheres, and must cut each of the *five* spheres at the same angle.

To ascertain how many solutions can be obtained to this problem, we must ascertain how many different pairs of such small circles can be found;* or, which is the same thing, we must find how many different pairs of planes of similitude

* I will add that I was indebted to a paper by Mr. R. J. Adcock in the "Analyst" for 1877, which impelled me to undertake a geometrical solution of this problem as a sequence of my former investigations. Mr. Adcock had given the equations for an analytical solution. It was followed by a solution of the same equations by Dr. Craig, of the Johns Hopkins University, which appeared in the number of the "Analyst" for January, 1880.

I was also indebted to Mr. Marcus Baker of the U. S. Coast Survey, who, in the number of the "Analyst" for July, 1877, gave out the question at my instance, as it had attracted our attention in Vol. 1st of Crelle, 1826. It was Steiner who proposed it in that number, without a solution. I have never been able to find Steiner's solution, if he ever gave one. I will add that after I made known to Mr. Baker this solution of the problem, we in the hunt for the number of solutions independently reached the same conclusion, that there must be six hundred and forty solutions.

It should be stated that all of this memoir, except the two last problems, were completed and sent to the Smithsonian Institute in January, 1860, from Fort Vancouver, Washington Territory, but the manuscript was burned in January, 1865, when the upper story of the Smithsonian building was on fire. The two last problems (from Article 31) were solved in 1878, and read in November of that year to the National Academy of Science at its meeting in New York.

can be obtained for the five given spheres. The five spheres A, B, C, D and E can be grouped in five sets of four each, as

1. A, B, C, D.
2. A, B, C, E.
3. A, B, D, E.
4. A, C, D, E.
5. B, C, D, E.

Take A, B, C, D, each of their 8 planes of similitude can be paired with each of the 8 planes of similitude of

	A, B, C, E	64 pairs.	
Also of	A, B, D, E	64	
"	A, C, D, E	64	
"	B, C, D, E	64	256

Each of A, B, C, E with each of the 8 planes of those after it in the above table, viz.

With	A, B, D, E	64	
	A, C, D, E	64	
	B, C, D, E	64	192

Each of A, B, D, E, with those of

A, C, D, E	64	
B, C, D, E	64	128
The eight of A, C, D, E		
with eight of B, C, D, E . . .	64	64
Grand total		640

Thus there are six hundred and forty different spheres which can be drawn cutting the five given spheres at the same angle.

As for Circles (Article 31) some of these intersections will be imaginary, but the sphere obtained graphically will be situated in a similar manner towards the given spheres.*

*Since writing the above my attention has been called to a paper by M. Darboux, in p. 323 of Vol. 1st, 2d series of *Annales de L'École Normale Supérieure*, Paris, 1872, upon "the relations between the groups of points of circles and of spheres in a plane and in space," in which will be found solutions of the leading questions in this paper.

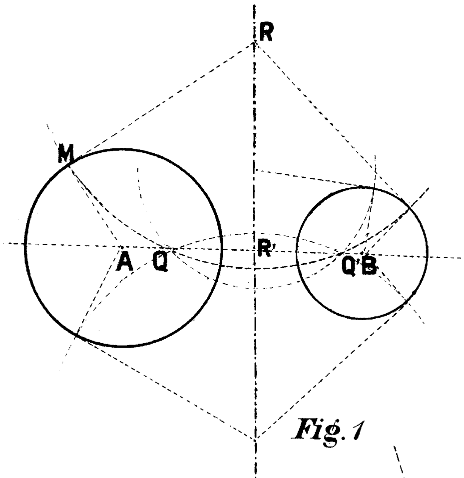


Fig. 1

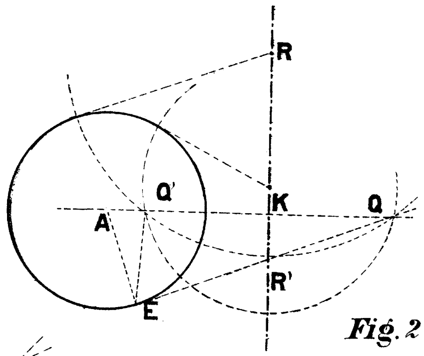


Fig. 2

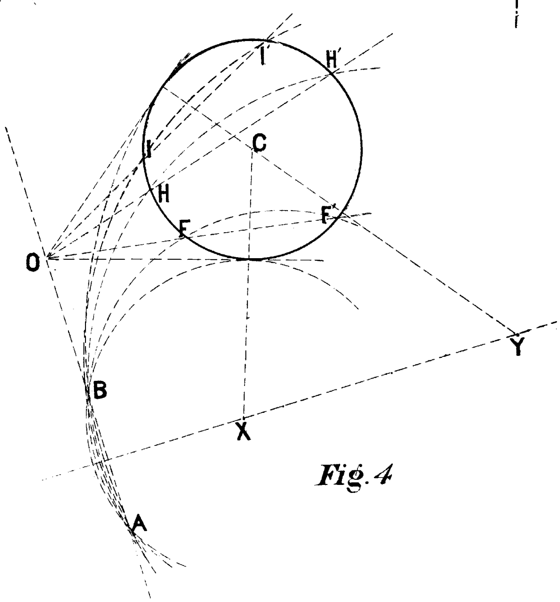


Fig. 4

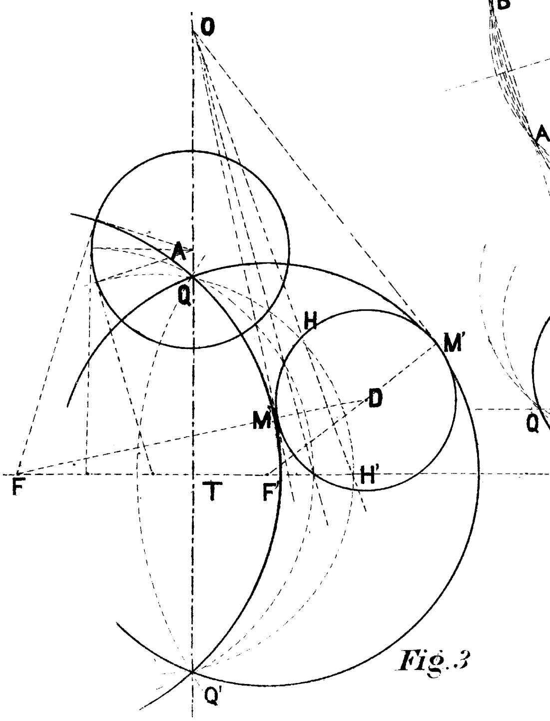


Fig. 3

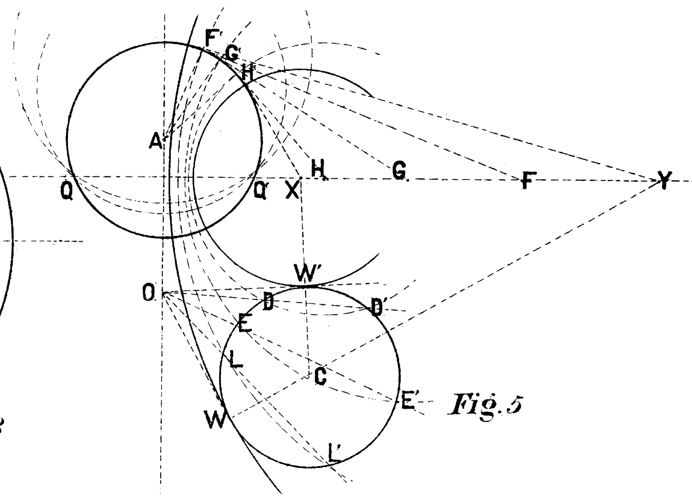


Fig. 5

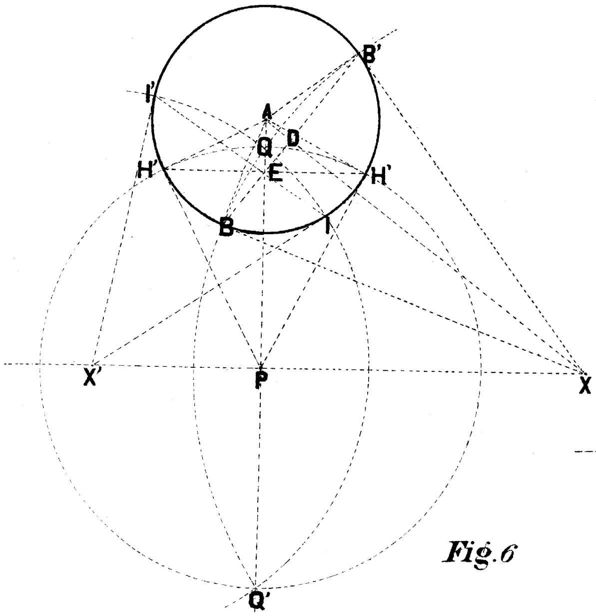


Fig. 6

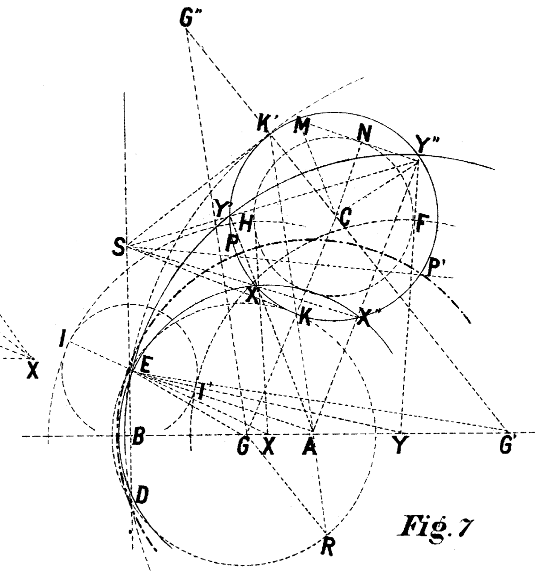


Fig. 7

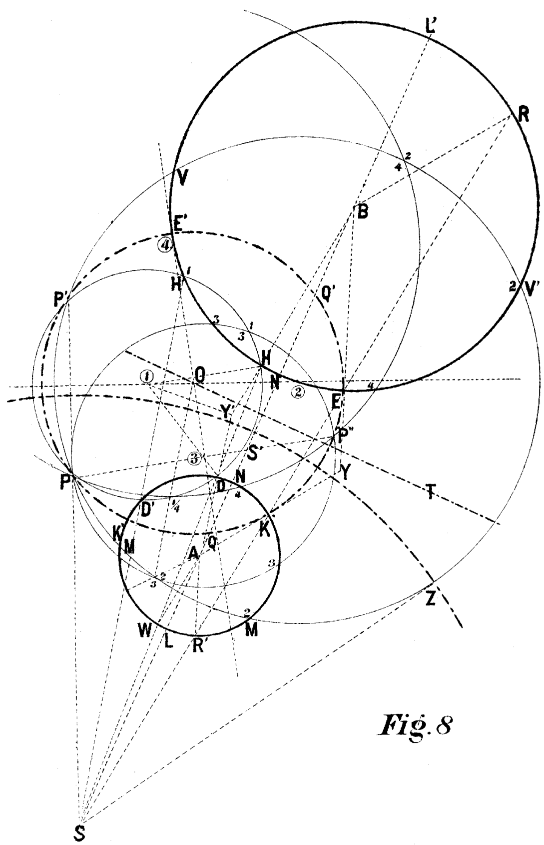


Fig. 8

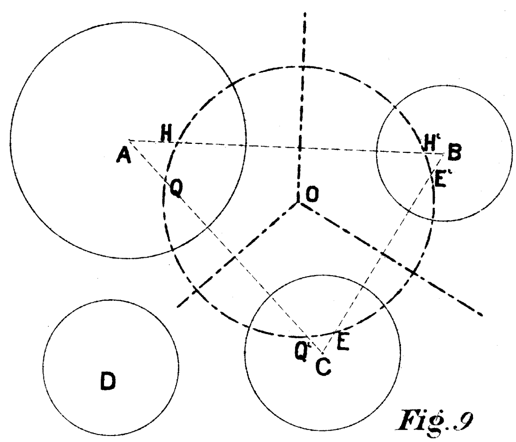


Fig. 9

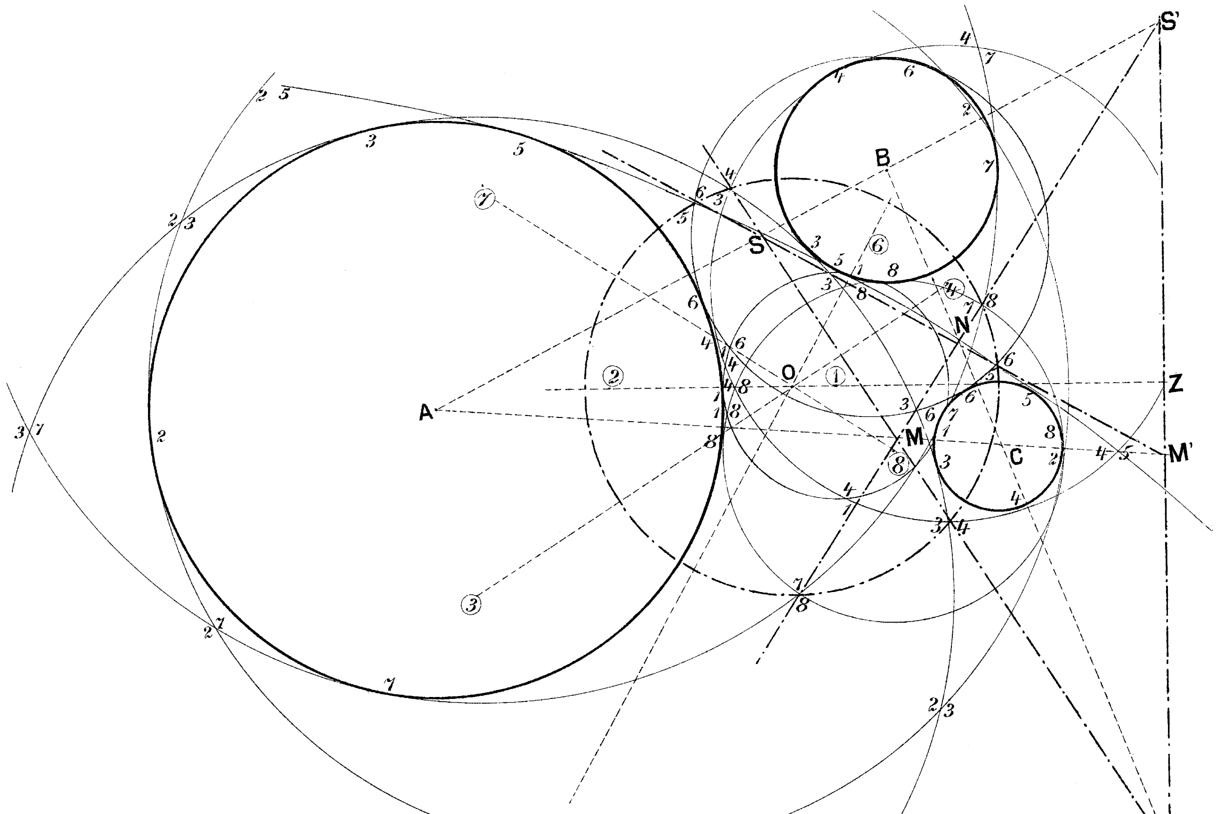
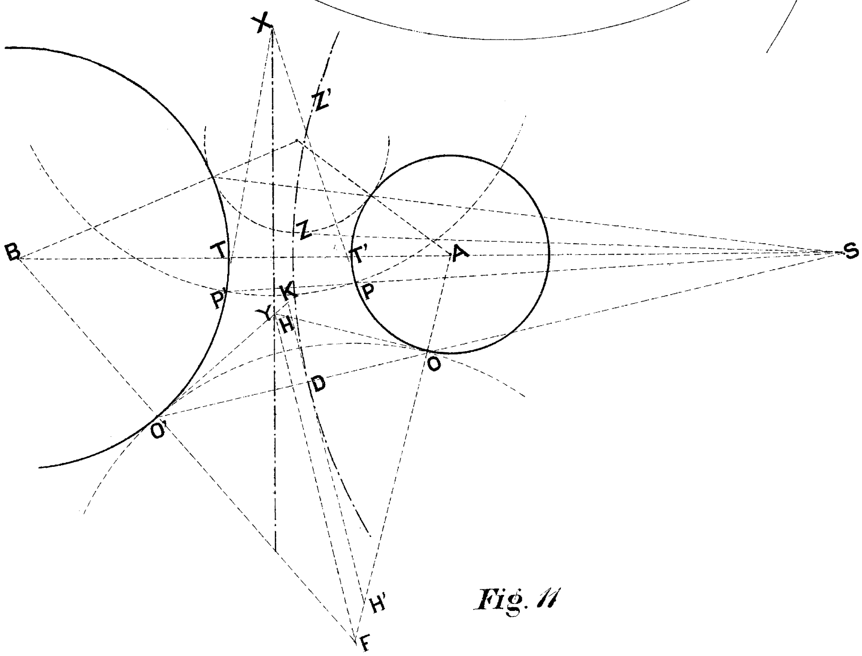


Fig. 10



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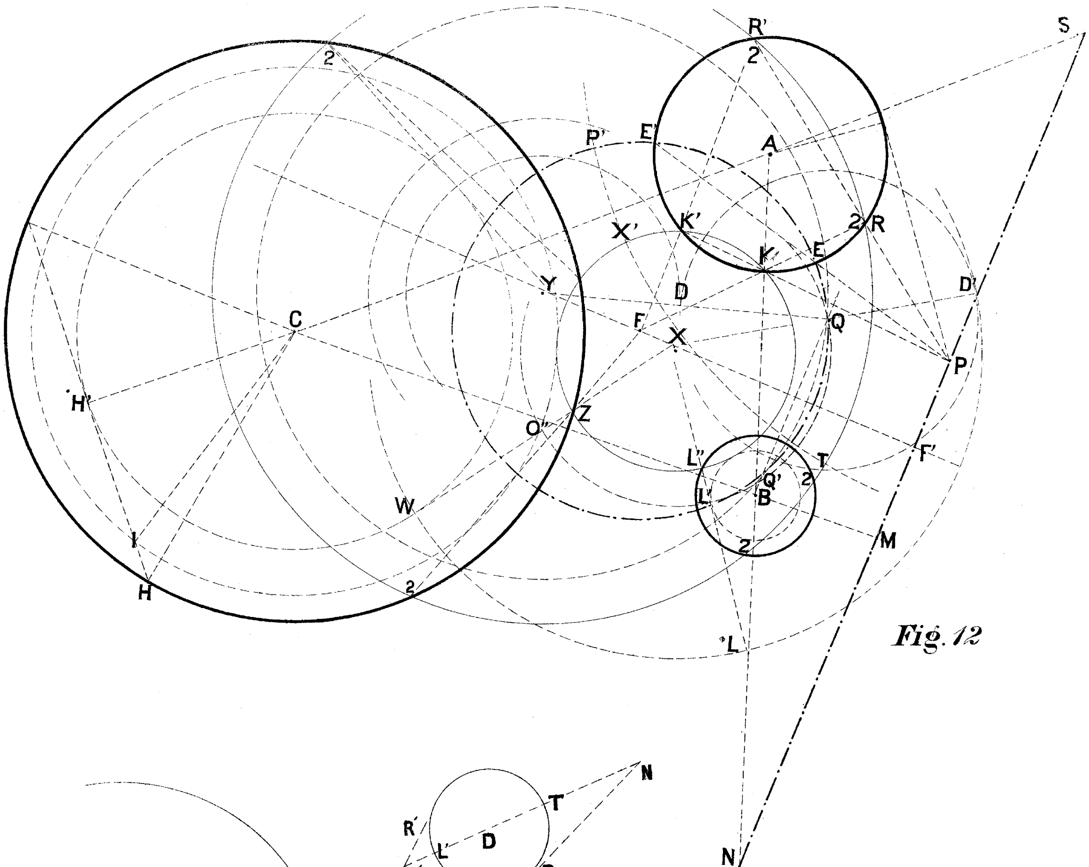
R'
S
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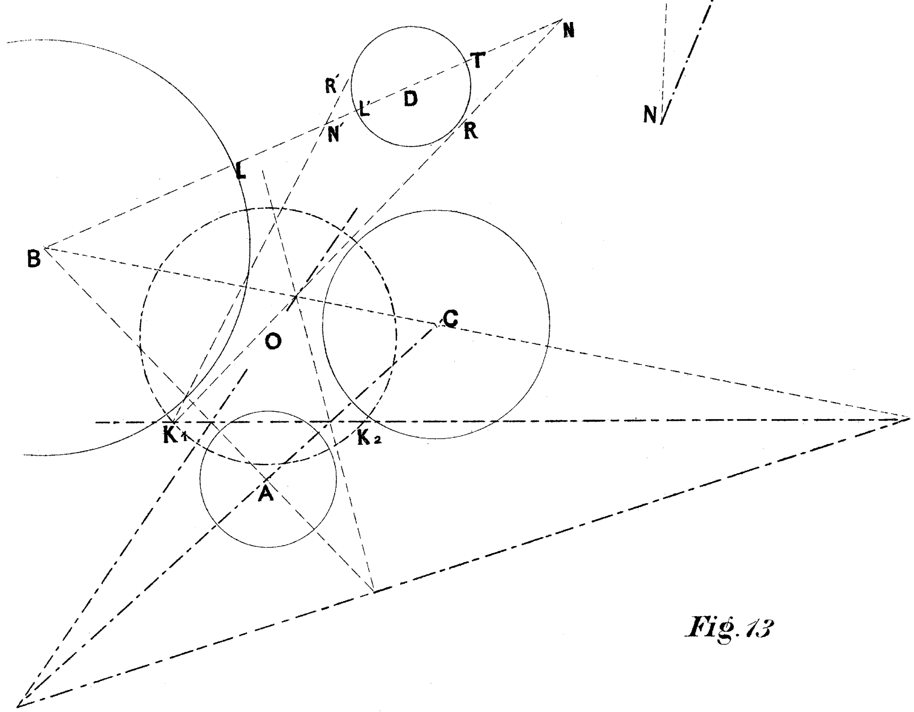
Fig. 12


N
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K₂
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Fig. 13